Linear Algebra
[KOMS119602] - 2022/2023

11.2 - Fundamental spaces: row, column, and null spaces

Dewi Sintiari

Computer Science Study Program
Universitas Pendidikan Ganesha

Week 11 (November 2022)

Row vectors and column vectors

Given an $m \times n$ matrix A:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} & \end{bmatrix}$$

- Row vector: vector formed from a row of A
- Column vector: vector formed from a column of A

Row vectors and column vectors

The row vectors of A are:

$$\begin{aligned}
\mathbf{r}_1 &= [a_{11} \;\; a_{12} \;\; \cdots \;\; a_{1n}] \\
\mathbf{r}_2 &= [a_{21} \;\; a_{22} \;\; \cdots \;\; a_{2n}] \\
\vdots &= \qquad \vdots \\
\mathbf{r}_m &= [a_{m1} \;\; a_{m2} \;\; \cdots \;\; a_{mn}]
\end{aligned}$$

The column vectors of A are:

$$\mathbf{c}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \; \mathbf{c}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \; \ldots, \; \mathbf{c}_1 = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Let A be an $(m \times n)$ matrix.

- The subspace of $\mathbb{R}^n$ formed by row vectors of A is called row space of matrix A.
- Subspace of $\mathbb{R}^m$ formed by column vectors of A is called column space of matrix A.
- The solution space of the homogeneous linear system $A\mathbf{x} = \mathbf{0}$ (which is a subspace of $\mathbb{R}^n$) is called null space of matrix A.

Relationship

Question 1. What relationships exist among the solutions of a linear system $A\mathbf{x} = \mathbf{b}$ and the row space, column space, and null space of the coefficient matrix A?

Question 2. What relationships exist among the row space, column space, and null space of a matrix?

Column space

Consider the system $A\mathbf{x} = \mathbf{b}$ where:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots a_{mn} & \end{bmatrix} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Let $\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n$ be the column vectors of A. The system can be written as:

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ \Leftrightarrow x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + \cdots + x_n\mathbf{c}_n &= \mathbf{b} \end{aligned}$$

Hence, the system has a solution if and only if $\mathbf{b}$ can be expressed as a linear combination of the column vectors of A.

Theorem

A system of linear equations $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathbf{b}$ is in the column space of A.

Example of column space

Given a linear system $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} -1 & 3 & 2 \\ 1 & 2 & -3 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -9 & -3 \end{bmatrix}$$

Show that $\mathbf{b}$ is in the column space of A by expressing it as a linear combination of the column vectors of A.

Solution:

Steps:

- Solve the system by Gaussian elimination:

$$x_1 = 2, \; x_2 = -1, \; x_3 = 3$$

- This yields:

$$2\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ -3 \end{bmatrix}$$

  i.e.,

$$x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3 = \mathbf{b}$$

Null space

Given matrix:

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & 1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix}$$

To determine the null space of A, solve the homogeneous linear system $A\mathbf{x} = \mathbf{0}$:

$$A = \begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & 1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the system by Gauss elimination, we obtain:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s - t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

The solution of the system can be written in matrix equation:

$$\mathbf{x} = s\mathbf{v}_1 + t\mathbf{v}_2$$

where $s, t \in \mathbb{R}$, $\mathbf{v}_1 = (-1, 1, 0, 0, 0)$ and $\mathbf{v}_2 = (-1, 0, -1, 0, 1)$.

Determine the basis of null space

Properties of row/column space and null space

Theorem

Elementary row operations do not change the **row space** *of a matrix.*

Theorem

Elementary row operations do not change the **null space** *of a matrix.*

How to determine the basis of row space, column space, and null space?

Let A be an $(m \times n)$ matrix. How to determine the basis of row space, column space, and null space of matrix A?

1. Perform elementary row operations to obtain the reduced-row echelon form matrix R;
2. The basis of the row space of A in all row vectors that contain leading 1 * of matrix R;
3. The basis of column space of A is all column vectors of matrix A that correspond with the column vector of matrix R that contains leading 1.

*Leading 1 is the leading entry in each nonzero row is 1

Intuition behind the algorithm

Example 1: determining the basis for row space and column space

Determine the basis of row space, column space, and null space of matrix:

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \sim ERO \sim \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

The basis of the row space is:

$$\mathbf{r}_1 = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \end{bmatrix}$$
$$\mathbf{r}_2 = \begin{bmatrix} 0 & 0 & 1 & 3 & -2 & -6 \end{bmatrix}$$
$$\mathbf{r}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 5 \end{bmatrix}$$

Example 1 (*cont.*)

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix} \leftrightarrow \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

So, the basis of the column space is:

$$\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ -1 \end{bmatrix} \quad \mathbf{c}_2 = \begin{bmatrix} 4 \\ 9 \\ 9 \\ -4 \end{bmatrix} \quad \mathbf{c}_3 = \begin{bmatrix} 5 \\ 8 \\ 9 \\ -5 \end{bmatrix}$$

Example 2: determining the basis of null space

To determine the basis of null space, solve the equation $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 0 \\ 2 & -6 & 9 & -1 & 8 & 2 & 0 \\ 2 & -6 & 9 & -1 & 9 & 7 & 0 \\ -1 & 3 & -4 & 2 & -5 & -4 & 0 \end{bmatrix} \sim ERO \sim \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 0 \\ 0 & 0 & 1 & 3 & -2 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The linear system correspond to the last augmented matrix is:

$$\begin{cases} x_1 - 3x_2 + 4x_3 - 2x_4 + 5x_5 + 4x_6 = 0 \\ x_3 + 3x_4 - 2x_5 - 6x_6 = 0 \\ x_5 + 5x_6 = 0 \end{cases}$$

from which we can extract the following:

$$\begin{aligned} x_5 &= -5x_6 \\ x_3 &= -3x_4 + 2x_5 + 6x_6 = -3x_4 + 2(-5x_6) + 6x_6 = -3x_4 - 4x_6 \\ x_1 &= -3x_2 - 4x_3 + 2x_4 - 5x_5 - 4x_6 \\ &= -3x_2 - 4(-3x_4 - 4x_6) + 2x_4 - 5(-5x_6) - 4x_6 \\ &= -3x_2 + 14x_4 + 22x_6 \end{aligned}$$

Example 2 (*cont.*)

Let $x_2 = r$, $x_4 = s$, and $x_6 = t$, then the solution of $A\mathbf{x} = \mathbf{0}$ is:

$$\begin{aligned} x_1 &= -3x_2 + 14x_4 + 22x_6 = -3r + 14s + 22t \\ x_3 &= -3x_4 - 4x_6 = -3s - 4t \\ x_5 &= -5t \end{aligned}$$

This can be written as vectors:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \text{-}3r + 14s + 22t \\ r \\ -3s - 4t \\ s \\ -5t \\ t \end{bmatrix} = \begin{bmatrix} \text{-}3r \\ r \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 14s \\ 0 \\ -3s \\ s \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 22t \\ 0 \\ -4t \\ 0 \\ -5t \\ t \end{bmatrix} = r \begin{bmatrix} \text{-}3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 14 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 22 \\ 0 \\ -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

The basis of the null space is:

$$\mathbf{v}_1 = (-3, 1, 0, 0, 0, 0), \ \mathbf{v}_2 = (14, 0, -3, 1, 0, 0), \ \mathbf{v}_3 = (22, 0, -4, 0, -5, 0)$$

Rank and Nullity

In Example 1, we found that the row space and column space of matrix:

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

both contain three vectors. Hence, they are both three-dimensional spaces.

Does this hold for other matrices?

Dimension of row space and column space

Theorem

The row space and the column space of a matrix A have the same dimension.

Proof.

- The elementary row operations do not change the dimension of the row space and column space of a matrix.
- Let R be any row echelon form of A, then:

$$\dim(\text{row space of } A) = \dim(\text{row space of } R)$$
$$\dim(\text{column space of } A = \dim(\text{column space of } R)$$

- $\dim(\text{row space of } R) =$ the number of nonzero rows in R; and
- $\dim(\text{column space of } R) =$ the number of leading 1's in R.

Since in R, the number of nonzero rows $=$ the number of leading 1's, hence $\dim(\text{row space of } A) = \dim(\text{column space of } A)$. □

Rank and nullity

The dimension of the row space (and column space) of a matrix A is called the rank of A, and denoted by $\text{rank}(A)$.

The dimension of the *null space* of A is called the nullity of A, and denoted by $\text{nullity}(A)$.

Theorem (Dimension Theorem for Matrices)

If A is a matrix with n columns, then:

$$rank(A) + nullity(A) = n$$

Example

Find the rank and nullity of the matrix (size (4×6):

$$A = \begin{bmatrix} -1 & 2 & 0 & 4 & 5 & -3 \\ 3 & -7 & 2 & 0 & 1 & 4 \\ 2 & -5 & 2 & 4 & 6 & 1 \\ 4 & -9 & 2 & -4 & -4 & 7 \end{bmatrix}$$

Solution:

- **Rank**

  The reduced row echelon form of A is (verify it!):

$$R = \begin{bmatrix} 1 & 0 & -4 & -28 & -37 & 13 \\ 0 & 1 & -2 & -12 & -16 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

  Since there are two rows with leading 1, then:

$$\dim(\text{row space of } A) = \dim(\text{column space of } A) = 2$$

Example (*cont.*)

- **Nullity**

  To find the nullity, solve the linear system: $A\mathbf{x} = \mathbf{0}$.

  From the reduced echelon form of A, we obtain the following linear system:

$$\begin{cases} x_1 \quad - 4x_3 - 28x_4 - 37x_5 + 13x_6 = 0 \\ \quad x_2 - 2x_3 - 12x_4 - 16x_5 + \ \ 5x_6 = 0 \end{cases}$$

  Solving these equations for the *leading variables* yields:

$$x_1 = 4x_3 + 28x_4 + 37x_5 - 13x_6$$
$$x_2 = 2x_3 + 12x_4 + 16x_5 - 5x_6$$

  So, the solution of the system is:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example (*cont.*)

Hence, the vectors:

$$\begin{bmatrix} 4 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 28 \\ 12 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 37 \\ 16 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -13 \\ -5 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

form a **basis** for the solution space, then:

$$\text{nullity}(A) = 4$$

Remark. Observed that:

$$\begin{aligned} \text{rank}(A) + \text{nullity}(A) &= n \\ 2 + 4 &= 6 \end{aligned}$$

Conclusion

Theorem

If A is an $(m \times n)$ matrix, then:

1. *$rank(A)$ = the number of leading variables in the general solution of $A\mathbf{x} = \mathbf{0}$.*
2. *$nullity(A)$ = the number of parameters in the general solution of $A\mathbf{x} = \mathbf{0}$.*

Exercise:

Find the rank and nullity of the matrix:

$$A = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 2 & -6 & 9 & -1 & 8 & 2 \\ 2 & -6 & 9 & -1 & 9 & 7 \\ -1 & 3 & -4 & 2 & -5 & -4 \end{bmatrix}$$

Solution of exercise

The reduced echelon form of the matrix is the following:

$$R = \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 \\ 0 & 0 & 1 & 3 & -2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three nonzero rows in the matrix, so $\text{rank}(A) = 3$.

By the "Dimension Theorem", $\text{nullity}(A) = n - \text{rank}(A) = 6 - 3 = 3$

Solution of exercise (*cont.*)

To prove that nullity$(A) = 5$, we solve the linear system: $A\mathbf{x} = \mathbf{0}$.

$$\begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 0 \\ 2 & -6 & 9 & -1 & 8 & 2 & 0 \\ 2 & -6 & 9 & -1 & 9 & 7 & 0 \\ -1 & 3 & -4 & 2 & -5 & -4 & 0 \end{bmatrix} \sim ERO \sim \begin{bmatrix} 1 & -3 & 4 & -2 & 5 & 4 & 0 \\ 0 & 0 & 1 & 3 & -2 & -6 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the reduced augmented matrix, we get the linear system:

$$\begin{cases} x_1 - 3x_2 + 4x_3 - 2x_4 + 5x_5 + 4x_6 = 0 \\ x_3 + 3x_4 - 2x_5 - 2x_6 = 0 \\ x_5 + 5x_6 = 0 \end{cases}$$

Solving the system for the leading 1's yields:

$$\begin{aligned} x_5 &= -5x_6 \\ x_3 &= -3x_4 - 8x_6 \\ x_1 &= 3x_2 + 14x_4 + 57x_6 \end{aligned}$$

Solution of exercise (*cont.*)

Hence, the solution of the system can be written as:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3r + 14s + 57t \\ s \\ -3s - 8t \\ s \\ -5t \\ t \end{bmatrix} = r \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 57 \\ 0 \\ -8 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

where $r, s, t \in \mathbb{R}$.

Hence, the basis of the null space of A is:

$$\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 57 \\ 0 \\ -8 \\ 0 \\ -5 \\ 1 \end{bmatrix} \right\}$$

which means that $\text{nullity}(A) = 3$.

Equivalent statements

If A is an $(n \times n)$ matrix, then the following statements are equivalent.

1. A is invertible.
2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. The reduced row echelon form of A is I_n.
4. A is expressible as a product of elementary matrices.
5. $A\mathbf{x} = \mathbf{0}$ is consistent for every $(n \times 1)$ matrix b.
6. $A\mathbf{x} = \mathbf{0}$ has exactly one solution for every $(n \times 1)$ matrix b.
7. $\det(A) \neq 0$.
8. The column vectors of A are linearly independent.
9. The row vectors of A are linearly independent.
10. The column vectors of A span $\mathbb{R}^n$.
11. The row vectors of A span $\mathbb{R}^n$.
12. The column vectors of A form a basis for $\mathbb{R}^n$.
13. The row vectors of A form a basis for $\mathbb{R}^n$.
14. A has rank n.
15. A has nullity 0.